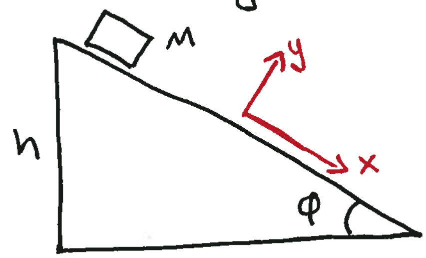
force diagram example:



find an equation for the position of the block with respect to time

① force analysis, free body diagram

$$f_x = mg\cos\varphi$$

② Newton's equation

$$f_x = ma$$
$$= m\ddot{x}$$

so

$$\ddot{x} = g\cos\varphi$$

in physics, sometimes a dot is used to represent a time derivative:

$$V = \frac{dx}{dt} = \dot{x}$$
$$a = \frac{d^2x}{dt^2} = \frac{d}{dt}\dot{x} = \ddot{x}$$

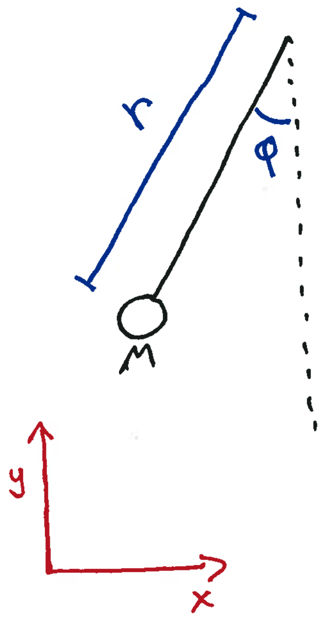
③ Integration

$$x = \iint \ddot{x}\, dt^2$$
$$= \frac{1}{2}g\cos\varphi\, t^2$$

This is a very simple problem. for more complex systems we need a better approach.

Introducing more complex systems

1) Ball on string pendulum



- Two coordinate systems.
- Each has "two" dimensions
- Each has the same information

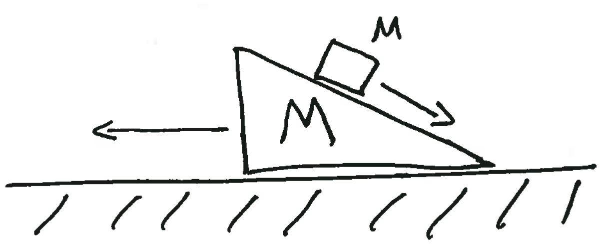
a) cartesian

b) Polar

- Look closely: 1 dimensional system, φ!

Can you write down $X(t)$? $\varphi(t)$?

2) Wedge sliding under a block, no friction

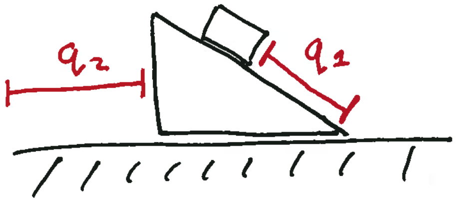


- complicated!
- 4 position dimensions
- can you solve it?

Can't solve this easily

Introducing the Lagrangian

1) generalized coordinates.
only the important quantities for energy and motion, etc.



only <u>2</u> dimensions instead of <u>4</u>

2) Kinetic Energy: $\frac{1}{2}mv^2$, $\frac{1}{2}I\omega^2$... T
translational, rotational

3) Potential energy: mgh, $\frac{1}{2}kx^2$, ... U
gravity, spring

T = kinetic ; U = Potential

The quantity

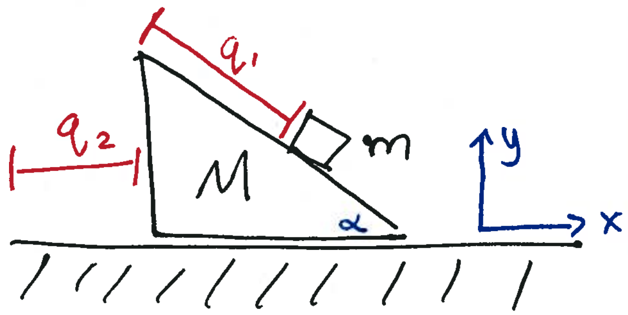
$$\boxed{\mathcal{L} = T - U}$$

And the equation

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)$$

give the equation of motion for a system. Let's look at an example

Example of Lagrangian



$$\boxed{\begin{aligned} x_m &= q_2 + q_1 \cos\alpha \\ y_m &= -q_1 \sin\alpha \\ x_M &= q_2 \end{aligned}}$$

① what is the kinetic energy T?

$$T_m = \frac{1}{2} m v^2 = \frac{1}{2} m (\dot{q}_1^2 + \dot{q}_2^2 + 2\dot{q}_1 \dot{q}_2 \cos\alpha)$$

(This $v^2 = \dot{x}_m^2 + \dot{y}_m^2$)

$$T_M = \frac{1}{2} M \dot{q}_2^2$$

so

$$T = T_m + T_M = \frac{1}{2}(M+m)\dot{q}_2^2 + \frac{1}{2} m (\dot{q}_1^2 + 2\dot{q}_1 \dot{q}_2 \cos\alpha)$$

② what is the potential energy?

$$U = mg y_m$$

$$= -mg y_m = -mg q_1 \sin\alpha$$

③ write down the Lagrangian $\mathcal{L}$

$$\mathcal{L} = T - U$$

$$= \boxed{\frac{1}{2}(M+m)\dot{q}_2^2 + \frac{1}{2} m (\dot{q}_1^2 + 2\dot{q}_1 \dot{q}_2 \cos\alpha)} + \boxed{mg q_1 \sin\alpha}$$

Lagrangian → Equation of motion

recall: $\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}$

In this example, we have <u>2</u> coordinates

$$x \longrightarrow q_1, q_2$$

So we have <u>2</u> equations

$$\left\{ \begin{aligned} \frac{\partial \mathcal{L}}{\partial q_1} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1} \\ \frac{\partial \mathcal{L}}{\partial q_2} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_2} \end{aligned} \right\}$$

don't get intimidated by these, they are usually <u>very simple equations</u>

Let's solve the q_1 equation first. Remember, these are partial derrivitives, so $\frac{\partial q}{\partial \dot{q}} = \frac{\partial \dot{q}}{\partial q} = 0$!

$$\left(\mathcal{L} = \tfrac{1}{2}(M+m)\dot{q}_2^2 + \tfrac{1}{2}m(\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2\cos\alpha) - mgq_1\sin\alpha\right)$$

Plug in:

$$mg\sin\alpha = \frac{d}{dt}\left(m[\dot{q}_1 + \dot{q}_2\cos\alpha]\right)$$

$$\boxed{mg\sin\alpha = m\ddot{q}_1 + m\ddot{q}_2\cos\alpha}$$

The other eqn for q_2 gives

$$\boxed{\ddot{q}_2(m+M) = m\ddot{q}_1\cos\alpha}$$

Algebra: $\boxed{\ddot{q}_1 = \dfrac{g\sin\alpha}{1 - \dfrac{m\cos^2\alpha}{M+m}}}$

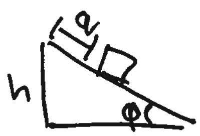
This is all we need to know about the system!

135 quiz!

1) write down the lagrangian for a ball in free fall

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 - mgx$$

2) write down the lagrangian for system we looked at earlier:



$$\mathcal{L} = \frac{1}{2} m \dot{q}^2 - mg\sin\alpha$$

3) Tricky: write Lagrangian for a Pendulum with moment of inertia I:

$$\mathcal{L} = \frac{1}{2} I \dot{\varphi}^2 + mg\frac{L}{2}\cos\varphi$$

<u>CM</u> $\frac{L}{2}$

Symmetry & Lagrangian

Symmetry can tell you a lot

"I have a rotationally symmetric object"

$\Downarrow$

$\boxed{\text{It's a ball}}$

Symmetry in mechanics:

- Rotational $\longleftrightarrow$ conservation of angular momentum
- Translation $\longleftrightarrow$ conserve momentum (linear)
- Time $\longleftrightarrow$ conserve energy

To say that the Lagrangian $\mathcal{L}$ is rotationally symmetric is to say

$$\boxed{\frac{\partial \mathcal{L}}{\partial \varphi} = 0}$$

This implies certain constraints on what $\mathcal{L}$ can be.

By picking symmetries, you determine the form of the Lagrangian.

Then the Lagrangian determines the behavior of the system.

The standard model Lagrangian

Symmetry:

4d poincare &

$SU(3)_c \times SU(2)_L \times U(1)_Y$

format:

$$\mathcal{L} = \mathcal{L}_{\text{kinetic}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{yukawa}}^{\text{(flavor, strong)}}$$

This is the most general form of Lagrangian given above symmetries

There is a lot of detail hidden in ~~these~~ this Lagrangian (which I don't understand)

However, this is a lagrangian like the ones we found before.

This Lagrangian specifies particle masses, interactions, etc. It describes

everything we know about particle physics.